Angular Momentum

Introduction

Angular momentum is classically defined as:

$$\mathbf L=\mathbf r\times\mathbf p$$

The quantum-mechanical operator is defined the same way, by just replacing the two quantities with their respective operators.

$$ \hat L_i =\hat x_j\hat p_k-\hat x_k\hat p_j =-i\hbar\left(x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j}\right) $$

The three components of angular momentum are not compatible quantities with each other, since they don't commute.

$$\left[\hat L_i,\hat L_j\right]=i\hbar\hat L_k\Leftrightarrow\mathbf L\times\mathbf L=i\hbar\mathbf L$$

In order to fully describe the angular momentum of a quantum-mechanical system we need another quantity, which would be compatible. This quantity is the magnitude of the total angular momentum squared, $\hat L^2=\left|\hat{\mathbf{L}}\right|^2$.

Indeed, this quantity commutes with all the components.

$$\left[\hat L^2,\hat L_i\right]=0$$

Completely describing the angular momentum of a quantum system we use two operators, $\hat L^2$ for the magnitude and $\hat L_z$ for its projection on the z axis, which is easier to play with in spherical coordinates.

Also, both of those operators commute with the Hamiltonian, and thus have common eigenfunctions. In the end, we have two eigenvalue problems.

$$\hat L^2\left|l,m\right>=\hbar^2\lambda\left|l,m\right>=\hbar^2l(l+1)\left|l,m\right>$$ $$\hat L_z\left|l,m\right>=\hbar\left|l,m\right>$$

The eigenvalue $\lambda=l(l+1)$ is written this way because it is easier to handle mathematically, as will be shown later on, without affecting the problem in any way. The quantum numbers l,m are integers and define the so called **Spherical Harmonics** $Y_{lm}(\theta,\phi)$, which are the angular part of the wavefunctions $\psi_{lm}(r,\theta,\phi)=R(r)\cdot Y_{lm}(\theta,\phi)$.

The existence of a central potential $V(\mathbf r)=V(-\mathbf r)$ causes the quantization of the radial part of the wavefunction. Together with spin they form a quartet of quantum numbers (n,l,m_l,m_s) which fully defines the state of a particle around the central potential (e.g. a *single* electron around a nucleus).

Calculating the Eigenvalues

The eigenvalue problem of $\hat L_z$ is similar to that of the Hamiltonian for the Quantum Harmonic Oscillator. We are looking for the *raising* and *lowering* operators $\hat L_\pm$ that will move us from the state $\left|l,m\right>$ to the state $\left|l,m\pm1\right>$.

The only difference is that l should not be affected, meaning that the eigenvalue equation for $\hat L^2$ will still hold. l represents the magnitude of the angular momentum, while m its projection on an axis. It is a natural requirement that any action on m, which corresponds to axes rotation, won't affect l in any way.

We start by factoring the magnitude:

$$L^2=L_x^2+L_y^2+L_z^2=L_z^2+\left(L_x+iL_y\right)\left(L_x-iL_y\right)$$

By comparing with the algebra of the harmonic oscillator (factoring $\hat a^\dagger\hat a$) we practically *guess* the existence of the operators:

$$\hat L_+=\hat L_x+i\hat L_y$$ $$\hat L_-=\hat L_x-i\hat L_y$$

It's obvious that $\hat L_\pm^\dagger=\hat L_\mp$. We now need to show that these two operators have the required properties:

- $\left[\hat L_\pm,\hat L^2\right]=0$
- $\left[\hat L_z,\hat L_\pm\right]=\pm\hbar\hat L_\pm$, which leads to $\hat L_\pm\left|l,m\right>\propto\left|l,m\pm1\right>$

First Property

The $\hat L_\pm$ operator is a linear combination of $\hat L_x, \hat L_y$ which commute with $\hat L^2$. So $\hat L_\pm$ commutes too.

$$ \left[\hat L_\pm,\hat L^2\right] =\left[\hat L_x,\hat L^2\right] \pm i\left[\hat L_y,\hat L^2\right] =0\pm i0=0 $$

Second Property

$$\begin{align} \left[\hat L_z,\hat L_\pm\right] &= \left[\hat L_z,\hat L_x\pm i\hat L_y\right] \\ &= \left[\hat L_z,\hat L_x\right]\pm i\left[\hat L_z,\hat L_y\right] \\ &= \left[\hat L_z,\hat L_x\right]\mp i\left[\hat L_y,\hat L_z\right] \\ &= i\hbar\left(\hat L_y\mp i\hat L_x\right) \\ &= \hbar\left(i\hat L_y\pm\hat L_x\right) \\ &= \pm\hbar\left(\hat L_x\pm i\hat L_y\right) \\ &= \pm\hbar\hat L_\pm \end{align}$$ The quantum number m is the eigenvalue of $\hat L_z$, so in order to see if $\hat L_\pm$ indeed takes us from state $\left|l,m\right>$ to state $\left|l,m\pm1\right>$ we can act with it on state $\hat L_\pm\left|l,m\right>$.

$$\begin{align} \hat L_z\left(\hat L_\pm\left|l,m\right>\right) &= \left(\hat L_z\hat L_\pm\right)\left|l,m\right> \\ &= \left(\hat L_\pm\hat L_z\pm\hbar\hat L_\pm\right)\left|l,m\right> \\ &= \hat L_\pm\left(\left(\hat L_z\pm\hbar\right)\left|l,m\right>\right) \\ &= \hat L_\pm\left(\hbar

m\pm\hbar\right)\left|l,m\right> \\ &= \hbar\left(m\pm1\right)\left(\hat L_\pm\left|l,m\right>\right) \end{align}$$ The eigenvalue is $\hbar\mu=\hbar\left(m\pm1\right)$, so the operator $\hat L_\pm$ indeed takes us from state $\left|l,m\right>$ to state $\left|,m\pm1\right>$.

Normalization

We have shown that there is a relation between the two states, $\hat L_\pm\left|l,m\right>\propto\left|l,m\pm1\right>$. In order to find an exact equation we can use the fact that wavefunctions are normalized.

But first we need to calculate two operators that show up later on during the process.

$$\begin{align} \hat L_-\hat L_+ &= \left(\hat L_x-i\hat L_y\right)\left(\hat L_x+i\hat L_y\right) = \hat L_x^2+\hat L_y^2+i\left(\hat L_x\hat L_y-\hat L_y\hat L_x\right) \\ &= \hat L^2-\hat L_z^2+i\left[\hat L_x,\hat L_y\right] = \hat L^2-\hat L_z^2+i\left(i\hbar\hat L_z\right) \\ &= \hat L^2-\hat L_z^2-\hbar\hat L_z = \hat L^2-\hat L_z\left(\hat L_z+\hbar\right) \end{align}$$ $$\begin{align} \hat L_+\hat L_- &= \left(\hat L_x+i\hat L_y\right)\left(\hat L_x-i\hat L_y\right) = \hat L_x^2+\hat L_y^2-i\left(\hat L_x\hat L_y-\hat L_y\hat L_x\right) \\ &= \hat L^2-\hat L_z^2-i\left[\hat L_x,\hat L_y\right] = \hat L^2-\hat L_z^2-i\left(i\hbar\hat L_z\right) \\ &= \hat L^2-\hat L_z^2+\hbar\hat L_z = \hat L^2-\hat L_z\left(\hat L_z-\hbar\right) \end{align}$$ If we write $\hat L_\pm\left|l,m\right>=c_{lm}^\pm\left|l,m\pm1\right>$, we then can find the constant by taking the square magnitudes, since $\left\|\left|l,m\pm1\right>\right\|^2=1$ by definition.

$$\begin{align} \left|c_{lm}^\pm\right|^2 &= \left\|\hat L_\pm\left|l,m\right>\right\|^2 \\ &= \left(\hat L_\pm\left|l,m\right>\right)^\dagger\left(\hat L_\pm\left|l,m\right>\right) \\ &= \left< l,m\left|\hat L_\mp\hat L_\pm\right|l,m\right> \\ &= \left< l,m\left|\hat L^2-\hat L_z\left(\hat L_z\pm\hbar\right)\right|l,m\right> \\ &= \left< l,m\left|\hbar^2\left[l(l+1)\right]-\hbar m\left(\hbar m\pm\hbar\right)\right|l,m\right> \\ &= \hbar^2\left[l(l+1)-m(m\pm1)\right]\left< l,m|l,m\right> \\ &= \hbar^2\left[l(l+1)-m(m\pm1)\right] \end{align}$$ We now have fully defined how the angular momentum raising and lower operators act on a state. $$\boxed{\hat L_+\left|l,m\right> = \hbar\sqrt{l(l+1)-m(m+1)}\left|l,m+1\right>}$$ $$\boxed{\hat L_-\left|l,m\right> = \hbar\sqrt{l(l+1)-m(m-1)}\left|l,m-1\right>}$$

Accepted Values

Constructing the Eigenfunctions

Solving the Eigenvalue Problem

Using Operators

Example calculations of different quantities

From:
https://wiki.plakatouras.com/ - **Personal Wiki**

Permanent link:
https://wiki.plakatouras.com/doku.php?id=notes:physics:quantum-mechanics:angular-momentum&rev=1790801068

Last update: **2026/09/30 20:44**