Angular Momentum

Introduction

Angular momentum is classically defined as:

$$\mathbf L=\mathbf r\times\mathbf p$$

The quantum-mechanical operator is defined the same way, by just replacing the two quantities with their respective operators.

$$ \hat L_i =\hat x_j\hat p_k-\hat x_k\hat p_j =-i\hbar\left(x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j}\right) $$

The three components of angular momentum are not compatible quantities with each other, since they don't commute.

$$\left[\hat L_i,\hat L_j\right]=i\hbar\hat L_k\Leftrightarrow\mathbf L\times\mathbf L=i\hbar\mathbf L$$

In order to fully describe the angular momentum of a quantum-mechanical system we need another quantity, which would be compatible. This quantity is the magnitude of the total angular momentum squared, $\hat L^2=\left|\hat{\mathbf{L}}\right|^2$.

Indeed, this quantity commutes with all the components.

$$\left[\hat L^2,\hat L_i\right]=0$$

Completely describing the angular momentum of a quantum system we use two operators, $\hat L^2$ for the magnitude and $\hat L_z$ for its projection on the z axis, which is easier to play with in spherical coordinates.

Also, both of those operators commute with the Hamiltonian, and thus have common eigenfunctions. In the end, we have two eigenvalue problems.

$$\hat L^2\left|l,m\right>=\hbar^2\lambda\left|l,m\right>=\hbar^2l(l+1)\left|l,m\right>$$ $$\hat L_z\left|l,m\right>=\hbar\left|l,m\right>$$

The eigenvalue $\lambda=l(l+1)$ is written this way because it is easier to handle mathematically, as will be shown later on, without affecting the problem in any way. The quantum numbers l,m are integers and define the so called **Spherical Harmonics** $Y_{lm}(\theta,\phi)$, which are the angular part of the wavefunctions $\psi_{lm}(r,\theta,\phi)=R(r)\cdot Y_{lm}(\theta,\phi)$.

The existence of a central potential $V(\mathbf r)=V(-\mathbf r)$ causes the quantization of the radial part of the wavefunction. Together with spin they form a quartet of quantum numbers (n,l,m_l,m_s) which fully defines the state of a particle around the central potential (e.g. a *single* electron around a nucleus).

Calculating the Eigenvalues

First Property

Second Property

Normalization

Accepted Values

Constructing the Eigenfunctions

Solving the Eigenvalue Problem

Using Operators

Example calculations of different quantities

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